

RELATIVISTIC FLUID DYNAMICS AS A HAMILTONIAN SYSTEM

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The equations of ideal relativistic fluid dynamics in the laboratory frame form a noncanonical hamiltonian system with the same Poisson bracket as for nonrelativistic fluids, but with dynamical variables and hamiltonian obtained via a regular deformation of their nonrelativistic counterparts.

Introduction. There are a number of fluid dynamical situations in which (special) relativistic effects are important. Such situations occur when either the velocity of the macroscopic motion is comparable to the velocity of light, or when there is sufficiently rapid microscopic motion. In astrophysics, for example, stars are commonly modeled as fluid bodies with relativistically high energy density and temperature. Relativistic fluid dynamics is also applied in certain models of free-electron LASERs [1] and particle beams.

The equations of relativistic hydrodynamics form a conservative dynamical system and, thus, are candidates for description within the framework of the hamiltonian formalism. Something approaching hamiltonian description of relativistic fluids has been given in ref. [2,3] from a constrained variational approach resulting in canonical Poisson brackets in terms of the so-called Clebsch potentials, some of which are unphysical. In nonrelativistic physics, during the last several years, considerable progress has been made in understanding hamiltonian structures of various continuum systems in terms of physical variables alone (see, e.g., refs. [4–15]). The resulting Poisson brackets in the physical variables are not canonical. Rather, they are associated with certain Lie algebras of semidirect product type [6–10,15]. Moreover, these noncanonical Poisson brackets are naturally connected with canonical brackets in Lagrange coordinates, under the classical map from the Lagrange description of fluids to the eulerian one

(even when the underlying lagrangian description is nonabelian and/or contains two-cocycles) [10]. Besides providing some general understanding of the underlying structure of classical fluid dynamical theories and serving as a guiding principle in the derivation of new fluid theories [9,15], these noncanonical Poisson brackets are also useful in practical applications. For example, these noncanonical brackets have recently been instrumental in deriving nonlinear stability conditions for nonrelativistic fluids [16].

It is natural to ask whether the noncanonical hamiltonian structures in the nonrelativistic case extend to the relativistic one, and if so, how do they extend? This raises the point that there are no guiding mathematical principles for such an extension, or even for the existence of such an extension. However, heuristic argument in favor of existence of such an extension is provided by the result of Bialynicki-Birula and Iwinski [17], who found the noncanonical Poisson brackets for a relativistic free fluid (“free” refers to noninteracting fluid particles streaming freely, without pressure forces). Remarkably, the noncanonical hamiltonian structure of the free relativistic fluid is identical in form to that (found seven years later!) for a nonrelativistic compressible fluid with pressure forces, in appropriately chosen physical variables. The main result of the present paper is that this form-invariance of the hamiltonian structure under relativisation carries over also for fluids with pressure interaction.

This result suggests that there exists an underlying principle of form-invariance upon relativisation of the hamiltonian structures of fluids and plasmas in interaction with additional fields. A direct proof of this conjecture would require verification for each additional interacting field (e.g., MHD, gravity, etc.)^{†1}. Here we identify the appropriate physical variables, hamiltonian functional, and Poisson bracket for the basic relativistic fluid description. In this case, the Poisson bracket retains the same form as for compressible, adiabatic, nonrelativistic fluids. Then we demonstrate that the proposed Poisson bracket and relativistic hamiltonian imply the correct relativistic fluid equations.

Nonrelativistic fluid. In ideal hydrodynamics, the physical variables are: ρ , mass density; \mathbf{M} , fluid momentum density; and either σ , entropy density, or $\eta = \sigma/\rho$, specific entropy. The fluid moves through euclidean space R^n with positions $x^i, i = 1, \dots, n$.

In the nonrelativistic case, the velocities v_i are related to momentum densities by

$$v_i = M_i/\rho. \quad (1)$$

The eulerian hydrodynamics equations are expressed as

$$\dot{M}_i = -(\rho^{-1} M_i M_j + \delta_{ij} p)_{,j}, \quad (2)$$

$$\dot{\rho} = -M_{j,j}, \quad (3)$$

$$\dot{\eta} = -\rho^{-1} \eta_{,j} M_j, \quad (4)$$

where the dot denotes partial time derivative $\partial/\partial t$ and we sum on repeated indices. Latin indices i, j run from 1 to n with $n = 3$ for the physical case, and subscript comma denotes partial derivatives with respect to the indicated variables. Eq. (2) is the hydrodynamic motion equation expressed in conservative form as the divergence of the nonrelativistic stress tensor. The fluid pressure p determined as a function of ρ and η from a prescribed relation (equation of state) for the specific internal energy $e(\rho, \eta)$, combined with the first law of thermodynamics

$$de = e_\rho d\rho + e_\eta d\eta = \rho^{-2} p d\rho + T d\eta,$$

^{†1} We have verified this conjecture for a relativistic multi-fluid plasma.

where T is temperature. Eq. (3) is the continuity equation and eq. (4) is the adiabatic condition for each fluid element.

The hydrodynamic system (2)–(4) can be expressed as a hamiltonian system $\dot{F} = \{H, F\}$ with the hamiltonian

$$H = \int [M^2/2\rho + \rho e(\rho, \eta)] d^n x \quad (5)$$

equal to the nonrelativistic energy. The Poisson bracket $\{F, G\}$ for functionals F and G is defined to be [4–6]

$$\begin{aligned} \{F, G\} = - \int d^n x & \left[\frac{\delta G}{\delta \rho} \partial_j \rho \frac{\delta F}{\delta M_j} - \frac{\delta G}{\delta \eta} \eta_{,j} \frac{\delta F}{\delta M_j} \right. \\ & \left. + \frac{\delta G}{\delta M_i} \left(\rho \partial_i \frac{\delta F}{\delta \rho} + \eta_{,i} \frac{\delta F}{\delta \eta} + (M_j \partial_i + \partial_j M_i) \frac{\delta F}{\delta M_j} \right) \right]. \end{aligned} \quad (6)$$

The hydrodynamic equations are then identical to $\dot{F} = \{H, F\}$, $F \in \{\rho, \sigma, \mathbf{M}\}$.

Remark. The quantity $\int \rho d^n x$ is in the kernel of the Poisson bracket (6), i.e., $\{\int \rho d^n x, F\} = 0, \forall F$, so H in (5) can be changed to $H' = H + \alpha \int \rho d^n x$ for any constant α without affecting the hydrodynamic equations. Other functionals in the kernel of Poisson bracket (6) are, for arbitrary functions Ψ, Φ of their indicated arguments,

$$\int \rho \Psi(\eta) d^n x, \quad \int \rho \Phi(\rho^{-1} \text{curl}(\mathbf{M}/\rho) \cdot \nabla \eta) d^3 x, \quad (7)$$

the latter of which is strictly three-dimensional. The quantity $\Omega = \rho^{-1} \text{curl}(\mathbf{M}/\rho) \cdot \nabla \eta$ is known as potential vorticity and the conserved functional $\int \rho \Phi(\Omega) d^3 x$ plays an important role in the nonlinear stability of ideal fluids [16].

Relativistic fluid. The equations of relativistic hydrodynamics in covariant form are [18]

$$T^\mu_{\nu,\mu} = 0, \quad (8)$$

$$(\rho u^\mu)_{,\mu} = 0, \quad (9)$$

where Greek indices μ, ν range over $0, 1, \dots, n, x^0 = ct$ being the real time coordinate. Thermodynamic quantities such as density ρ are evaluated in the proper frame of a moving fluid element. The metric tensor is given by the expression $-d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ for the

proper time interval and $g_{\mu\nu}$ has signature +2. The equations of motion are contained in (8) which expresses conservation of energy and momentum, while (9) is the relativistic continuity equation.

The components of the energy-momentum tensor $T_{\mu\nu}$ in three-dimensional form are given in [18], where it is also shown that (8) and (9) imply the adiabatic equation

$$u^\mu(\sigma/\rho)_{,\mu} = \frac{d(\sigma/\rho)}{d\tau} = 0,$$

in which the derivative $d/d\tau$ is taken along the world line of the fluid element concerned.

For the purposes of hamiltonian formulation we express these equations as a dynamical system in the laboratory frame. Thus,

$$\partial \tilde{M}_i / \partial t = -(\tilde{M}_i v_j)_{,j} - p_{,i}, \quad (10)$$

$$\partial \tilde{\rho} / \partial t = -(\tilde{\rho} v_j)_{,j}, \quad (11)$$

$$\partial \tilde{E} / \partial t = -[(\tilde{E} + p) v_j]_{,j}, \quad (12)$$

and, consequently,

$$\partial \tilde{\eta} / \partial t = -\tilde{\eta}_{,j} v_j, \quad (13)$$

where $\tilde{\rho}$, \tilde{M} , \tilde{E} and $\tilde{\eta}$ are, respectively, the mass density, momentum density, energy density (including rest energy), and specific entropy in the laboratory reference frame. The velocity of matter relative to the laboratory frame is denoted by \mathbf{v} , and p is the pressure in the rest frame. The laboratory frame quantities are related to rest-frame quantities by

$$\tilde{\rho} = \gamma \rho, \quad (14)$$

$$\tilde{\mathbf{M}} = \gamma^2 (\rho c^2 + \rho e + p) \mathbf{v} / c^2 = \gamma^2 \rho \mathbf{v} w, \quad (15)$$

$$\tilde{E} = \gamma^2 (\rho c^2 + \rho e + p) - p = \gamma^2 \rho c^2 w - p, \quad (16)$$

$$\tilde{\eta} = \eta, \quad (17)$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ and $w = 1 + (e + p/\rho)/c^2$.

Notice that when $c^{-2} \rightarrow 0$, the relativistic equations (10), (11), and (13) limit to the nonrelativistic equations (2), (3), and (4), respectively. Now let us state the main result of the present work.

The relativistic equations (10), (11), and (13) follow from the same hamiltonian structure (5), re-expressed in relativistic variables \tilde{M} , $\tilde{\rho}$, $\tilde{\eta}$, with hamiltonian $\tilde{H} = \int (\tilde{E} - c^2 \tilde{\rho}) d^n x$, where \tilde{E} is given by formula (16).

Proof of this is based on the observation that the variational derivatives of the hamiltonian \tilde{H} can be expressed as

$$\delta \tilde{H} / \delta \tilde{M}_k = v_k, \quad (18)$$

$$\delta \tilde{H} / \delta \tilde{\rho} = c^2 [w(1 - v^2/c^2)^{1/2} - 1], \quad (19)$$

$$\delta \tilde{H} / \delta \eta = c^2 \rho w_{,\eta} - p_{,\eta}, \quad (20)$$

while the hamiltonian \tilde{H} itself can be rewritten as

$$\tilde{H} = \int \{c^2 \{[M^2/c^2 + (\tilde{\rho} w)^2]^{1/2} - \tilde{\rho}\} - p\} d^n x. \quad (21)$$

By substituting identities (18)–(20) into equations $\dot{F} = \{\tilde{H}, F\}$ in terms of the Poisson bracket (6) with variables (M_i, ρ, η) replaced by variables $(\tilde{M}_i, \tilde{\rho}, \tilde{\eta})$, one immediately recovers the relativistic motion equations (10), (11), and (13).

Remarks. (A) Just as relativistic equations (10), (11), and (13) tend to their nonrelativistic counterparts, so do the relativistic variational derivatives (18)–(20), and the hamiltonian (21), as well. Thus, the relativistic theory of fluids is a *regular, structure preserving deformation*, with parameter c^{-2} , of the nonrelativistic theory. The desire to keep all the formulae regular in c^{-2} motivates choosing the hamiltonian \tilde{H} to be the total energy $\int \tilde{E} d^n x$ minus the rest mass energy in the laboratory frame $c^2 \int \tilde{\rho} d^n x$: since $\int \tilde{\rho} d^n x$ is the kernel of the hamiltonian structure, this choice of \tilde{H} does not affect the motion equations.

(B) Because of form-invariance of the hamiltonian structure, the counterparts of functionals (7) give conservation laws for relativistic fluid dynamics, (10)–(13). These conservation laws may be used in the stability analysis of relativistic fluid dynamics, by the method employed in [16].

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